Growth-optimal Crypto-investment

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Motivation

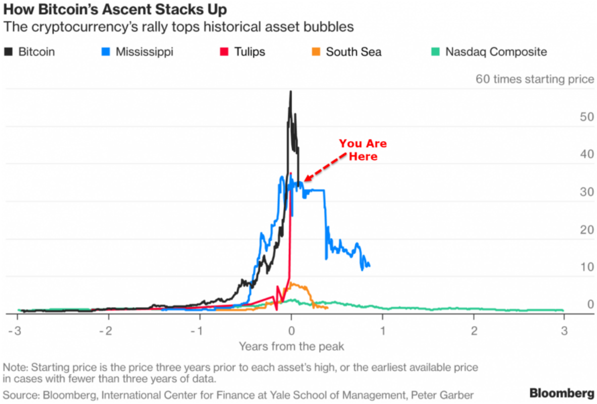

Figure: Bitcoin growth in comparison to past financial bubbles

Wealth equation

- Wealth for investment horizon $T \in \mathbb{N}^+$ and $k \in \mathbb{N}^+$ assets, given initial wealth $W_0 \in \mathbb{R}^+$

$$
\begin{aligned}
W_T(f_t) &= W_0 \prod_{t=1}^{T} \left(1 + \sum_{j=1}^{k} f_{j,t} X_{j,t} \right) \\
&= W_0 \prod_{t=1}^{T} \left(1 + f_t^\top X_t \right) = W_0 \prod_{t=1}^{T} \left\{ f_t^\top \exp(\widetilde{X}_t) \right\}
\end{aligned}
\tag{1}
$$

- Discrete / log returns $X_t = [X_1, \dots, X_{k-1}, X_r]^\top$ / $\widetilde{X}_t$
- Risk free rate $X_k = X_r \in \mathbb{R}$
- Investment fractions $f_t = [f_1, \dots, f_{k-1}, f_r]^\top$

Markowitz

- One risky asset (Bitcoin) and one risk-free asset, $k = 2$
- Markowitz optimization (two-stage investment process)

$$f^* = \operatorname*{argmax}_{f \in \mathbb{R}^2} \mathsf{E}\{W_T(f)\} \tag{2}$$

gives under $\mathsf{E}\, X > X_r$

$$f^* = [\infty, -\infty], \tag{3}$$

- But: For the multi-stage investment process (Thorp, 1971)

$$P\{W_T(f^*) = 0\} \to 1 \tag{4}$$

Kelly

- Kelly optimization (multi-stage investment process)

$$f^* = \operatorname*{argmax}_{f \in \mathbb{R}^2} \mathsf{E}\left[\log\left\{W_T(f)\right\}\right]. \tag{5}$$

- Myopic log-optimal strategy $\Lambda^* = [f^*, \dots, f^*]$
- Significantly different strategy Λ

$$\mathsf{E}\left\{\log W_T(\Lambda^*)\right\} - \mathsf{E}\left\{\log W_T(\Lambda)\right\} \longrightarrow \infty, \tag{6}$$

- Kelly investor dominates asymptotically (Breiman, 1961)

$$\lim_{T\to\infty} \frac{W_T(\Lambda^*)}{W_T(\Lambda)} \xrightarrow{a.s.} \infty \tag{7}$$

Markowitz vs. Kelly

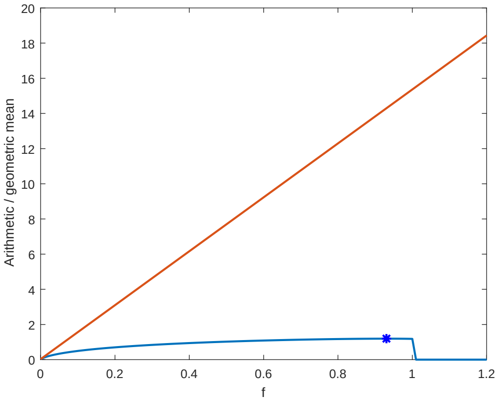

Figure: Arithmetic and geometric mean maximization

Data

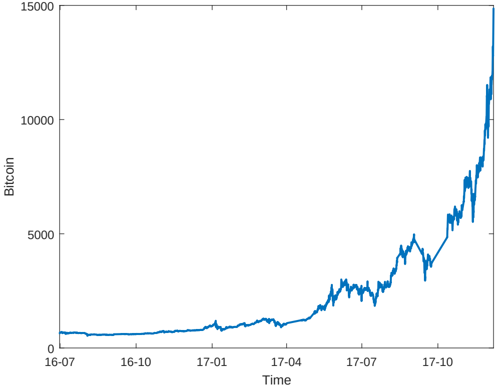

Figure: High-frequency data-set of Bitcoin from 07-2016 till 11-2017

Top 10 tail events within 5 minutes

	Surge	Drawdown
1	18.48	−22.35
2	13.04	−15.12
3	11.45	−11.00
4	8.83	−10.69
5	7.32	−8.92
6	6.72	−7.80
7	6.69	−7.71
8	5.82	−7.68
9	5.70	−7.41
10	5.11	−5.51

Table: Top ten surges and drawdowns in the 5-minute frequency (in %)

Heavy Tails

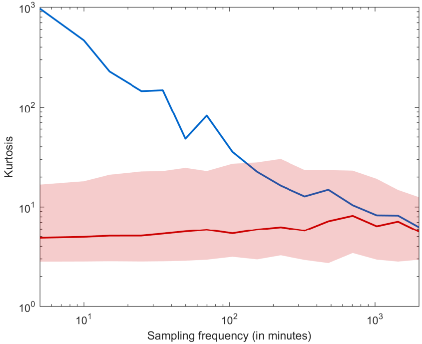

Figure: Sample kurtosis over sampling frequencies with bootstrapped sample kurtosis (red)

Heavy Tails

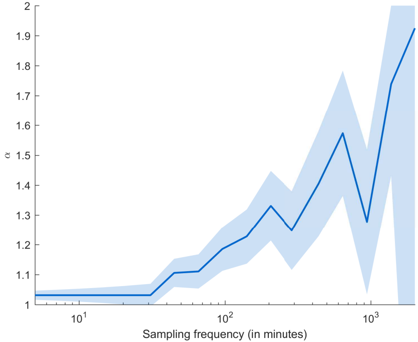

Figure: Stability exponent α over sampling frequencies

Kelly under Gaussianity (α=2)

▸ Closed-form solution

Parameters
$X_r = 0.01$
$\alpha = 2$
$\mathrm{E}\,\widetilde{X}_T = 1.18$
$\mathrm{E}\,X_T = 15.32$

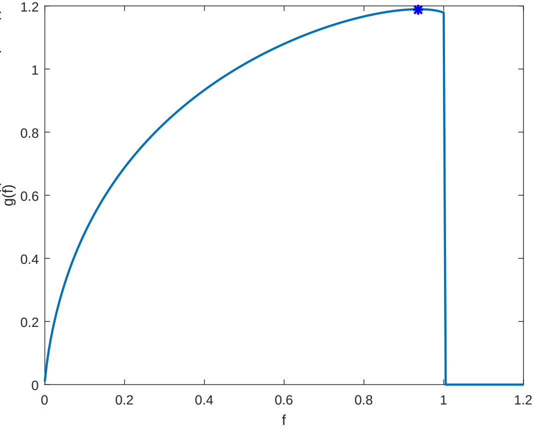

Figure: Optimal growth portfolio

Kelly under $\alpha - stability$

Parameters
$X_r = 0.01$
$\alpha = 2$
$[2 : 0.2 : 1]$
$\mathrm{E}\,\widetilde{X}_T = 1.18$
$\mathrm{E}\,X_T = 15.32$

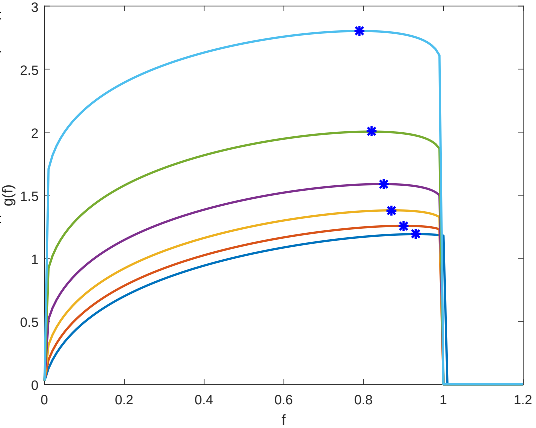

Figure: Optimal growth portfolio

Kelly under α − *stability*

Parameters
$X_r = 0.01$
$\alpha = 2$
$[2 : 0.2 : 1]$
$\mathrm{E}\,\widetilde{X}_T = 0$
$\mathrm{E}\,X_T = 4$

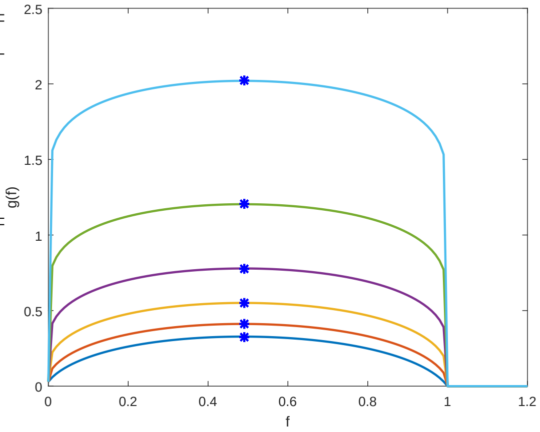

Figure: Optimal growth portfolio

Volatility induced growth

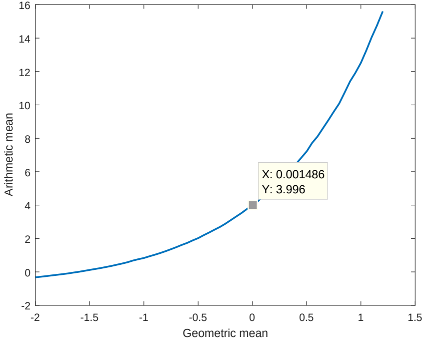

Figure: Geometric and arithmetic returns for Bitcoin

Kelly under $\alpha - \textit{stability}$

Parameters
$X_r = 0.01$
$\alpha = 2$
$\mathsf{E}\,\widetilde{X}_T = -1.43$
$\mathsf{E}\,X_T = 0.2$

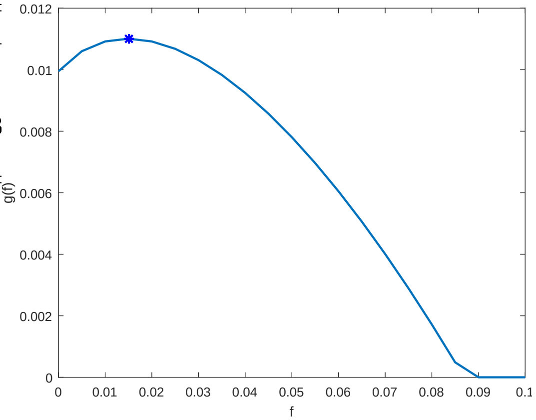

Figure: Optimal growth portfolio

Kelly under $\alpha - stability$

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$X_r = 0.01$
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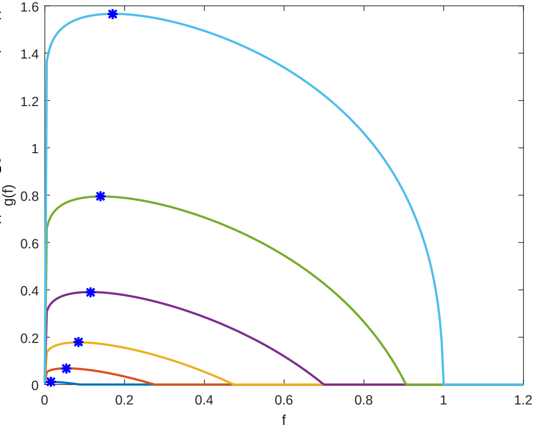

Figure: Optimal growth portfolio

Volatility induced growth

- Modelling log-returns $\widetilde{X}_t \sim \mathrm{N}(\widetilde{\mu}, \widetilde{\sigma})$ under Gaussianity ($\alpha = 2$)
- "Even if the growth rates of the individual securities all have mean zero, the value of a fixed-mix portfolio tends to infinity with probability one." (Dempster, Evstigneev and Schenk-Hoppe; 2006)
- Transformation to discrete returns for portfolios

$$X_t = \exp\left(\widetilde{X}_t\right) \sim \log \mathrm{N}(\mu, \sigma), \tag{8}$$

$$\mu = \exp\left(\widetilde{\mu} + \frac{\widetilde{\sigma}^2}{2}\right) \tag{9}$$

Stability induced growth

- Modelling log-returns $\widetilde{X}_t \sim \mathrm{S}(\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma}, \widetilde{\delta})$ under α-stability
- For $1 < \alpha < 2$, growth can be driven by stability
- Transformation to discrete returns for portfolios

$$X_t = \exp\left(\widetilde{X}_t\right) \sim \log \mathrm{S}(\alpha, \beta, \gamma, \delta), \quad (10)$$

$$\delta = \exp\left(\widetilde{\delta} + \widetilde{\gamma}^2 + g(\widetilde{\alpha})\right) \quad (11)$$

Parameter importance

- Gaussianity (Chopra and Ziemba, 2001)
  - $\mu \succ \sigma \succ \rho$
- Stability
  - $\delta \succ \gamma \overset{?}{\succ} \alpha \succ \rho$

Investment table, $\alpha = 2$

- Annual investment horizon

μ / $\widetilde{\sigma}$	100	140	180	220	260
5	2.59	0.75	0.20	0.05	0.01
10	5.98	1.94	0.58	0.15	0.03
20	13.63	4.56	1.54	0.50	0.13
50	37.55	14.10	5.25	1.85	0.59
100	71.30	30.42	12.53	4.97	1.72

Table: Optimal investment fractions given location and scale (in %)

Investment surface, $\alpha = 2$

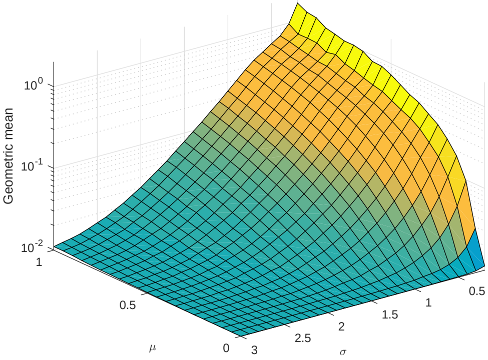

Figure: Optimal investment fractions given location and scale (in %)

Outlook

- Investigation of closed form solution for Kelly under α-stability
- Mean representation for log-stable distributions
- Relative portfolio importance of the underlying parameters

Finite variance (one-dimensional)

Presentation

- $X \sim F$ with $\mathrm{E}(X) = \mu$ and $\mathrm{Var}(X) = \sigma^2$
- Return of the risk free asset $r > 0$
- Wealth given investment fractions and restriction $\sum_{j=1}^{k} f_j = 1$

$$W(f) = W_0 \{1 + (1 - f)r + fX\} \tag{12}$$

$$= W_0 \{1 + r + f(X - r)\} \tag{13}$$

Finite variance (one-dimensional)

- Maximize

$$g(f) = \mathsf{E}\{\log W_T(f)\} \quad = \mathsf{E}\{G(f)\} = \mathsf{E}\log\{W_T(f)/W_0\} \tag{14}$$

- Wealth after n periods

$$W_T(f) = W_0 \prod_{t=1}^{T} \{1 + r + f(X_t - r)\} \tag{15}$$

- Taylor expansion of

$$\mathsf{E}\left[\log\left\{\frac{W_T(f)}{W_0}\right\}\right] = \mathsf{E}\left[\sum_{t=1}^{T} \log\{1 + r + f(X_t - r)\}\right] \tag{16}$$

Finite variance (one-dimensional)

- Given $\log(1+x) = x - \frac{x^2}{2} + \cdots$

$$\log\left\{1 + r + f(X - r)\right\} = r + f(X - r) - \frac{\{r + f(X - r)\}^2}{2} + \cdots \tag{17}$$

$$\approx r + f(X - r) - \frac{X^2 f^2}{2} \tag{18}$$

- Taking sum and expectation

$$\mathrm{E}\left[\sum_{t=1}^{T} \log\left\{1 + r + f(X_t - r)\right\}\right] \approx r + f(\mu_n - r) - \frac{\sigma_n^2 f^2}{2} \tag{19}$$

- Myopia: taking $\sum_{t=1}^{T} X_t$ has no impact on the solution

Finite variance (one-dimensional)

- Result of the Taylor expansion

$$g(f) = r + f(\mu - r) - \sigma^2 f^2/2 + \mathcal{O}(n^{-1/2}). \qquad (20)$$

- For $n \longrightarrow \infty$, $\mathcal{O}(n^{-1/2}) \longrightarrow 0$

$$g_\infty(f) = r + f(\mu - r) - \sigma^2 f^2/2. \qquad (21)$$

- Differentiating $g(f)$ according to f

$$\frac{\partial g_\infty(f)}{\partial f} = \mu - r - \sigma^2 f = 0 \leadsto f^* = \frac{\mu - r}{\sigma^2} = \sigma^{-1}\mathrm{MPR} \qquad (22)$$

- Betting the optimal fraction f^* leads to growth rate

$$g_\infty(f^*) = \frac{(\mu - r)^2}{2\sigma^2} + r. \qquad (23)$$

- $g_\infty(f)$ is parabolic around f^* with range $0 \leq f^* \leq 2f^*$

For Further Reading

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